BRANCHING OF STEADY TRAVELING-WAVE STATES OF A VISCOUS LIQUID FILM

Yu. Ya. Trifonov and O. Yu. Tsvelodub

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Shkadov [1] has derived a model system of equations describing the behavior of long-wavelength disturbances on a vertically flowing liquid film at moderate flow rates:

$$\frac{\partial q}{\partial t}+1{,}2\frac{\partial}{\partial x}\left(\frac{q^2}{h}\right)=-\frac{3\nu q}{h^2}+gh+\frac{\sigma}{\rho}h\frac{\partial^3 h}{\partial x^3},\ \frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=0. \tag{1}$$

Here q is the instantaneous mass flow rate of liquid in the cross section x, h is the instantaneous thickness of the film, g is the free-fall acceleration, σ is the coefficient of surface tension, ν is the kinematic viscosity coefficient, and ρ is the density of the liquid. It has been shown [2-4] that this system has two families of solutions in the form of steady-state traveling waves $h = h_0(\xi)$, $q = q_0(\xi)$, $\xi = x - ct$, which are quantitatively in good agreement with experiment [5] (c is the phase velocity). Nearly steady-state waves are observed experimentally only in a finite initial part of the flow, after which they evolve and break up, so that in order to understand this evolution of the flow, it is necessary to investigate the stability and bifurcations of the solutions obtained for the system (1).

The stability of the first family of waves (which branches off from the plane-parallel flow) was first analyzed [6] for Re ≳ 30. The stability of slightly nonlinear solutions of the equation describing the wave regimes for Re ≲ 1 is discussed in [7]. Others have systematically investigated the stability of both families [8, 9] and period-doubling bifurcation of the first family of waves [10]. In the present article we carry out a more general bifurcation analysis of the waves of this family and investigate new families of steady-state waves generated in the course of nonlinear evolution of unstable disturbances. Since the method described in [8, 9] can also be used in conjunction with stability analysis for the purpose of bifurcation analysis, it is essential to recall fundamental aspects of the formulation of the stability problem.

Transforming the system (1) to dimensionless form at the outset by analogy with [3, 4], making the substitutions

$$h(\xi, t)=h_0(\xi)+h'(\xi, t),\ q(\xi, t)=q_0(\xi)+q'(\xi, t), \tag{2}$$

in it, and linearizing the result for stability analysis of the solutions $h_0(\xi)$ and $q_0(\xi)$, we obtain a system of linear partial differential equations with ξ-periodic coefficients, whose solutions can be represented in the form

$$\begin{pmatrix} h(\xi, t)\\ q(\xi, t)\end{pmatrix}=e^{-\gamma t}\begin{pmatrix} h_1(\xi)\\ q_1(\xi)\end{pmatrix}+\text{c.c.} \tag{3}$$

(c.c. = complex conjugate).

The system of ordinary differential equations for $h_1(\xi)$ and $q_1(\xi)$ has solutions that are bounded for all ξ, which, according to the Floquet theorem, have the form

$$\begin{pmatrix} h_1\\ q_1\end{pmatrix}=e^{i\alpha Q\xi}\begin{pmatrix} \varphi(\xi)\\ \psi(\xi)\end{pmatrix}, \tag{4}$$

where φ and ψ are periodic functions having the same period as $h_0(\xi)$ and $q_0(\xi)$, and Q is a real parameter. After all the substitutions have been made in order to determine the spectrum of eigenvalues γ and the corresponding eigenfunctions φ, ψ, we arrive at the system

$$A\psi+B\frac{d\psi}{d\xi}+P\varphi-D\frac{d\varphi}{d\xi}-9i\alpha Qh_0\frac{d^2\varphi}{d\xi^2}-3h_0\frac{d^3\varphi}{d\xi^3}=\gamma\psi, \tag{5}$$
$$i\alpha Q\psi+\frac{d\psi}{d\xi}-i\alpha Qc\varphi-c\frac{d\varphi}{d\xi}=\gamma\varphi.$$

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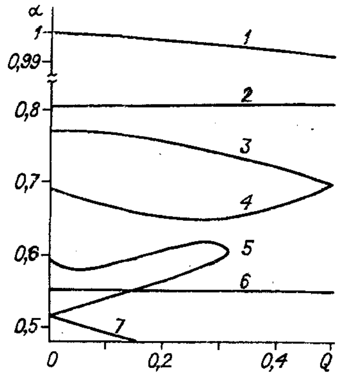

Fig. 1

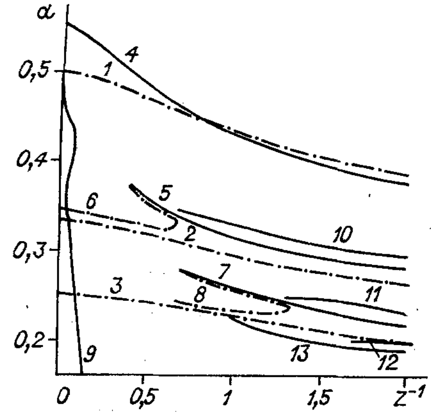

Fig. 2

Here

$$A = 2.4\frac{d}{d\xi}\left(\frac{q_0}{h_0}\right) + \frac{Z}{h_0^2} + i\alpha Q\left(2.4\frac{q_0}{h_0} - c\right);\ B = 2.4\frac{q_0}{h_0} - c;$$

$$P = -\left(1.2\frac{d}{d\xi}\left(\frac{q_0^2}{h_0^2}\right) + F + 2Z\frac{q_0}{h_0^3} + 3\frac{d^3h_0}{d\xi^3}\right) - 1.2i\alpha Q\frac{q_0^2}{h_0^2} + 3i\alpha^3Q^3h_0;$$

$$D = 1.2\frac{q_0^2}{h_0^2} - 9i\alpha^2Q^2h_0;$$

Z and F are parameters of the flow: $Z = (3\mathrm{We}/\mathrm{Re}^2)^{1/2}$, $F = (\mathrm{We}/3\mathrm{Fr}^2)^{1/2}$; $\mathrm{Re} = \langle q_0\rangle/\nu$ is the Reynolds number, $\mathrm{We} = \sigma\langle h_0\rangle/(\rho\langle q_0\rangle^2)$ is the Weber number, $\mathrm{Fr} = \langle q_0\rangle^2/(g\langle h_0\rangle^3)$ is the Froude number, and the angle brackets signify the average value with respect to the wavelength.

We have thus reduced the stability analyisis of the steady wave states $h_0(\xi)$ and $q_0(\xi)$ to the investigation of the spectrum of eigenvalues γ for various values of Q for which the system (5) has periodic solutions with the same period λ. The wave is stable if all $\mathrm{Re}(\gamma) \geqslant 0$ for any Q. Equation (4) clearly indicates that it is sufficient to limit the discussion to the variation of Q in any unit interval, say [−0.5, 0.5]. Taking the complex conjugate of Eq. (5), we readily verify that $\gamma(Q) = \bar{\gamma}(-Q)$. It is therefore sufficient to consider the solutions (5) for $0 \leqslant Q \leqslant 0.5$. It follows from Eqs. (2)-(4) that if the real part of some eigenvalue vanishes at a certain point (α, Q), a new wave state will branch off from the initial state. The generation of unsteady [if $\mathrm{Im}(\gamma) \neq 0$] as well as new steady [if $\mathrm{Im}(\gamma) = 0$] states is possible in this case. If Q = P/r is a rational number, a ξ-periodic state with $\alpha_{new} = \alpha/r$ is formed. If Q is an irrational number, a biperiodic state is generated.

The variables used here are such that for any values of the parameter Z periodic wave states of the first family branch off from the plane-parallel flow for the wave number $\alpha = 1$. This family is continued for any Z into the domain of small values of α and goes over to negative solitons in the limit $\alpha \to 0$ [2-4, 10]. Below, when speaking of new solutions that bifurcate from this family, we shall not recall each time that a solution belonging to the generating family itself also exist for these same values of α and Z.

Thus, in the space of the parameters Z, α, Q the steady states branch off at points located on the surfaces

$$\gamma(\alpha, Z, Q) = 0. \tag{6}$$

The cross sections of the first three surfaces (6) (the first surface has larger wave numbers) in the plane Z = 10 are represented by curves 1, 2, and 6 in Fig. 1. Unsteady states set in at the points of curves 3-5, 7 where $\mathrm{Re}(\gamma) = 0$ and $\mathrm{Im}(\gamma) \neq 0$. We note that for this value of Z waves of the first family are stable under disturbances having the same period (Q = 0) in the interval $0.518 \leqslant \alpha \leqslant 1$, and the domain of stability under disturbances with $Q \neq 0$ lies between curves 2 and 3. Clearly, the interval of stability under all disturbances is $-0.765 \lesssim \alpha \lesssim 0.82$.

We confine the present discussion to steady-state waves exclusively. For Fig. 1, new steady traveling-wave states with the largest wave numbers arise when the solutions branch off from curve 1 at points with Q = 1/2, 1/3, 1/4. These largest wave numbers are represented by curves 1-3, respectively, in Fig. 2 for various values of Z.

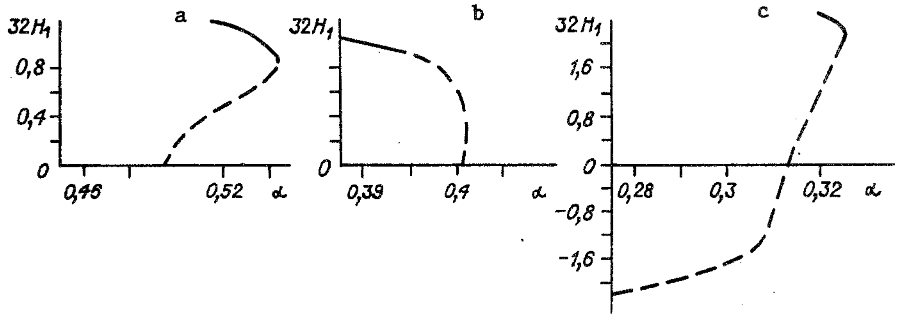

Fig. 3

Figure 3 affords a better picture of the branching process. It shows the amplitudes of the first harmonic of the generated wave as a function of α with increasing distance from the bifurcation point. For Fig. 3a, b ($Z^{-1} = 0.1$ and $Z^{-1} = 1.6$) the bifurcation points lie on curve 1 of Fig. 2, and for Fig. 3c ($Z^{-1} = 0.6$) they lie on curve 2. The bifurcation on curve 1 is one-sided. For all the investigated values of Z ($Z^{-1} \leqslant 2$) the solution is continued at first in the direction of large α and then, attaining the turning line (which is not shown in Fig. 2), it moves into the domain of smaller values of α. At the instant of generation, this family is unstable even under disturbances having the same periodicity (Q = 0). The nature of the stability under such disturbances changes when curve 4 is reached. The solutions corresponding to such instability are represented by the dashed parts of the curves in Fig. 3.

In an investigation of period-doubling bifurcations [10] the generating solution of the first family was chosen analytically in the form of the sum of the first two harmonics. The determination of the bifurcation points was reduced to finding the real roots of a quadratic equation. It was found in this approach (adapting to our notation) that the bifurcation equation does not have any real roots for $Z^{-1} > Z_*^{-1} \sim 1$, and branching of the initial solution does not take place. The discrepancy with our calculations is clearly attributable to the fact that the generating solution is not very accurately represented by two harmonics in this domain of the parameters.

In contrast with curve 1, which originates from the line of the first surface (6) $\gamma(\alpha, Z, 1/2) = 0$, curve 2 has a bifurcation that is two-sided with respect to α. This situation is also typical of the other curves (including curve 3 in Fig. 2) corresponding to rational numbers Q = P/r.

The new solutions obtained in the neighborhood of curves 1-3 are continued by continuity to the entire investigated domain of variation of the parameters $\alpha \geqslant 0.15$; $0.5 < Z < 100$. Calculations indicate the existence of a complicated interrelationship between these solutions. They form a multifolded and multisheeted surface on the plane of the parameters α, Z^{-1}. For example, if a solution originating on curve 1 is continued by continuity with the values of Z fixed, it can be continued for any Z to the smallest values of α used in the given calculations. For values of Z that are not common to curves 5 and 6 ($Z^{-1} \not\approx 0.4$-0.68) we obtain one solution for each α. The solution forms a fold between curves 5 and 6 in the domain $0.4 \lesssim Z^{-1} \lesssim 0.68$ and, moving downward as α is decreased, reaches curve 6, then returns along curve 5, and once again enters the domain of small α.

If we now fix α in the domain of existence of curve 5 ($0.284 \lesssim \alpha \lesssim 0.373$) and begin to move along the solution from $Z^{-1} < 0.4$ in the direction of larger values of Z^{-1}, we find that if our line α = const intersects curve 6, we stop at the same solution, having passed the above-described fold (curve 5-curve 6), but now, increasing α and Z^{-1}, we can come back to curve 1.

If, on the other hand, we move along α = const without intersecting curve 6, curve 5 is still the turning line, but now the solution is continued only to curve 2. Consequently, the solution generated by curve 1 transforms continuously into one of two bifurcation solutions originating from the first family with the parameter Q = 1/3 along curve 2, namely the one that branches off in the direction of larger α for a fixed value of Z.

If the solution branching off from curve 2 in the direction of larger α for $Z^{-1} < 0.68$ is continued by continuity for Z = const, we find that it turns upon reaching curve 6 and can be continued to small values of α. The nature of the turning from the curve is illustrated, in particular, by the behavior of the solution in the neighborhood of $\alpha \approx 0.325$ in Fig. 3.

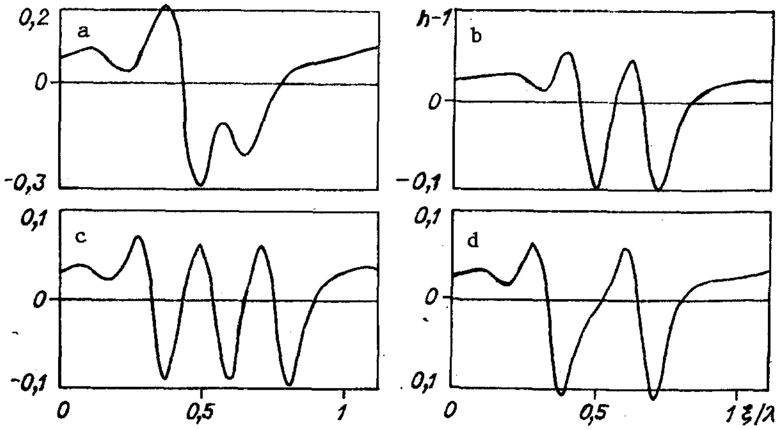

Fig. 4

It is interesting to note that if we move along the solution after turning from curve 6, increasing α and Z^{-1}, but without intersecting this curve a second time, we can return once again to curve 1. However, if we move from curve 2 for $Z^{-1} > 0.68$ until we reach curve 5, we turn in the direction of smaller α. Now the situation is largely similar to what happens in the branching from curve 1. This solution is also continued for any Z and has a fold in the domain $0.75 \leqslant Z^{-1} \leqslant 1.33$ between curves 7 and 8. Curve 7 is similar to curve 5: If we approach it from below, then after turning from it we enter a fold and, upon reflection from curve 8, continue to move along the same solution or go over continuously into one of the two solutions bifurcating from the first family with Q = 1/4 along curve 3.

The second solution originating from curve 2 is at once continued monotonically into the domain of small α for all the investigated values of Z^{-1}. Figure 4a shows the wave profile of this solution with α = 0.2 for $Z^{-1} = 0.6$. The second solution of curve 3 behaves similarly.

Consequently, for all points of curve 2 the solution branching off in the direction of larger α merges, although in a complicated way, with the solution branching off from curve 1. Three solutions exist at once in the domain bounded by curves 5, 6, and 2, which is a unique combination of a fold with a cut.

The domain bounded by curves 7, 8, and 3 also represents a similar fold–cut. We note that five solutions exist for α and Z in this domain. Three of them are analogous to the three preceding the fold and are obtained by continuation downward with respect to α from curve 5 and by branching from curve 3 upward with respect to α with subsequent turning at curves 6 and 8; the other two solutions are continued from curves 1 and 2 downward with respect to α.

Thus, even for the first of the surfaces (6) the branchings along the curves corresponding to the first maximum rational Q in the interval [0, 0.5] result in the formation of a whole set of new steady traveling-wave solutions, which are interrelated in a complicated way. This complication increases as α is decreased, because, on the one hand, the bifurcation lines cluster together more (cf. the distances between curves 1 and 2 and between curves 2 and 3 in Fig. 2), and the fold–cuts become narrower; on the other hand, the number of solutions increases rapidly. Moreover, other surfaces (6) begin to emerge for smaller values of α, and each one has its own complex set of solutions generated from it.

The wave profiles of the families branching off from the first family from the second surface (6) along curve 2 in Fig. 1 for Q = 1/2 and Q = 1/3 are shown in Fig. 4b and 4c, respectively, and the profiles corresponding to the third surface and motion along curve 6 in Fig. 1 for Q = 1/2 is shown in Fig. 4d (α = 0.2 and Z = 10 for all of them). This wave number is already quite close to zero; the profiles change very slightly with a further decrease in α, and for the most part only the fraction of nearly horizontal segments (h = const) increases. It is therefore reasonable to expect these solutions to go over to negative solitons with two or three troughs, respectively, in the limit α → 0.

Although it would be impossible to carry out a comprehensive analysis of all the solutions, it can be stated with a certain measure of caution that distinct (in the sense of stability) solutions are generated on the first bifurcation surface (6). They have quite a broad range of wave numbers that are stable under disturbances with Q = 0. For example, α

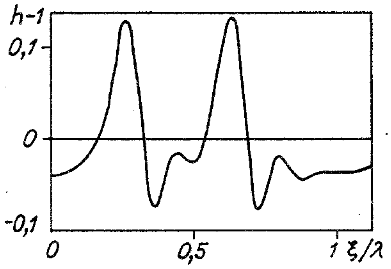

Fig. 5

lies between curves 4 and 9 in Fig. 2, and in the domains after the folds it lies between curves 4 and 10, 5 and 11, 7 and 12, and below curve 13, respectively. The distinction of this set of solutions is also expressed in the fact that some of the solutions have been shown, e.g., in [2-4], to be quantitatively in good agreement with the experimental data. They are the solutions that go over to single-hump solitons in the limit with respect to α, and they also exist in the domain of stability under all possible disturbances [8, 9]. Families that originate from other surfaces (6), judging from sample calculations, are unstable under disturbances having the same period for all the investigated values of α.

The above-described, rather complex hierarchy of solutions bifurcating from the first family is still not complete, because they are accompanied by the existence of surfaces (6) on which one of the eigenvalues γ crosses zero and new solutions emerge.

Figure 5 shows the profile of a wave of one of the families generated as a result of such a secondary bifurcation ($\alpha = 0.2$, $Z = 10$). This solution evidently has a limit in the form of a double-hump soliton. Multiple-hump solitons have been obtained [11] for the equation describing disturbances on a film in the case Re $\lesssim 1$. It is clear from the foregoing discussion that triple and higher-multiple bifurcations also exist.

Steady-state traveling waves partition the space of all periodic solutions into domains exhibiting different behavior. Our investigation demonstrates the exceedingly complex structure of the steady-state solutions of Eq. (1). Without knowing it, therefore, we cannot investigate the evolution of periodic disturbances with any kind of completeness, because for definite values of α and Z small variations in the initial data will eventually induce large disparities between the solutions. It is also clear that the presence of a large number of unstable steady-state solutions for sufficiently small α will, with a high degree of probability, impart stochastic behavior to the disturbances for any Z (and, accordingly, for any Re).

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